Polynomial algebras of parabolic invariants as modules over the Dickson algebra.*

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Abstract

A free module basis for the ring of upper triangular invariants over the Dickson algebra has been given firstly by Campbell and Hughes [1] and later by the author [3] using different methods. The analogue case for the ring of Borel invariants as a module over parabolic invariants has also been studied [4]. We extend the method employed in [3] to provide free bases of parabolic rings of invariants over the Dickson algebra for particular families of groups and a basis for any group consisting of two blocks. The multiplicative transfer which applies in mod-p cohomology is also studied between appropriate rings of invariants.

1 Introduction

Let $G = GL_n$, B_n, or U_n be the general linear group, the Borel subgroup, and the upper triangular subgroup with 1's on the diagonal, respectively. G acts as usual on V, an n-dimensional $\mathbb{F}_p$-vector space. We write $\mathbb{F}_p[V]$ for the ring of polynomial functions on $\overline{\mathbb{F}_p} \otimes_{\mathbb{F}_p} V$, where $\overline{\mathbb{F}_p}$ is an algebraic closure of $\mathbb{F}_p$.

$$\mathbb{F}_p[V] = \mathbb{F}_p[y_1, \cdots, y_n]$$

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Here, $V^* = \langle y_1, \cdots y_n \rangle$ and $\mathbb{F}_p[V]$ is a graded polynomial algebra with $|y_i| = 2$ (for topological reasons) or 1, if $p = 2$. The group G acts on $\mathbb{F}_p[V]$ via $(gf)(u) = f(g^{-1}u)$.

The Dickson algebra $D_n := \mathbb{F}_p[V]^{GL(n,\mathbb{F}_p)}$ plays a fundamental role in modular invariant theory of finite groups and serves as a computational tool in algebraic topology. If G is a finite group acting on a finite dimensional vector space V over the field $\mathbb{F}_p$, the ring $\mathbb{F}_p[V]^G$ is a finite extension of D_n.

It is well known that $\mathbb{F}_p[V]^{U_n}$, $\mathbb{F}_p[V]^{P(n_1,n-n_1)}$ and D_n are all polynomial algebras. Moreover, D_n serves as a homogeneous system of parameters and in fact both $\mathbb{F}_p[V]^{U_n}$ and $\mathbb{F}_p[V]^{P(n_1,n-n_1)}$ are free D_n-modules.

A free basis has been given for $\mathbb{F}_p[V]^{U_n}$ as a module over D_n ([1] and [3]) and $\mathbb{F}_p[V]^{P(n_1,n-n_1)}$ [4]. Here $P(n_1, n-n_1)$ is a parabolic subgroup of GL_n. In this work, we study the case $\mathbb{F}_p[V]^{P(n_1,n-n_1)}$ as a free module over D_n for some families of natural numbers $(n_1, n-n_1)$. The problem is focussed on the numbers: n and n_1. Only for certain families of pairs (n, n_1), the situation is completely analogous to the ones above, Proposition 27, but in general it is different than the previous ones and more complicated depending on the divisibility between natural numbers and certain relations between them. For the later case, we provide appropriate formulas (in section 2) which serve as a central tool to construct a free basis in Proposition 29. For the general case a module basis for $\mathbb{F}_p[V]^{P(n_1,n-n_1)}$ over D_n is given in Theorem 31:

Theorem 31 *The set* $B' = \left\{ \prod_{i=0}^{n_1-1} d_{n_1,i}^{m_i} \,|\, 0 \le m_i \le A_i \right\}$ *is a module basis for* $\mathbb{F}_p(n_1, n_2)$ *over* D_n.

Here the bounds A_i shall be defined in definition 17 and depend on n and n_1.

The interested reader can extend our method to obtain a free basis for particular choices using our formulas provided in section two and the preliminary subsection of section three.

Besides our invariant theoretic interest, we are motivated by topological properties of the transfer between cohomology of certain subgroups of the symmetric group. Namely, we are interested in multiplicative properties of the transfer studied by Kuhn and Priddy, because of their application in stable homotopy theory [6]. The transfer between parabolic subgroups is studied in the last section. Theorem 41 asserts that the natural map between $\mathbb{F}_p[V]^{B_n}$ and $\mathbb{F}_p(N, n)$ equals the transfer map and the analogue of Kuhn and Priddy result is expressed in Theorem 44:

Theorem 44 *Let* $\tau^* : \mathbb{F}_p[V]^G \to D_n$ *be the transfer map, where* $G = U_n$ *or* B_n. *Let* $x, y \in \mathbb{F}_p[V]^{U_n}$ *or* $\mathbb{F}_p[V]^{B_n}$. *Then* $\tau^*(xy) = \tau^*(x)\tau^*(y)$ *for all* y *iff* $x \in D_n$.

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We finish this introduction by recalling some well known Theorems relevant to the Dickson algebra.

Let $\sum_{p^n}$ be the symmetric group acting on V by permutations and ${\mathbb{F}_p}^n$ its subgroup consisting of all translations. Then $Aut({\mathbb{F}_p}^n) \cong GL_n$ and its Weyl subgroup, $W_{\sum_{p^n}}({\mathbb{F}_p}^n) \cong GL_n$, acts on V^* as follows:

$$(a_{i,j})y_k := \sum_i a_{i,k}y_i$$

We repeat some classical results from the literature. First the Dickson algebra, D_n, is described. Let

$$h_i = \prod_{a \in \langle y_1, \cdots, y_{i-1} \rangle} (y_i - a), \quad L_n = \prod h_i, \quad d_{n,i} = \frac{\begin{vmatrix} y_1 & \cdots & y_n \\ \vdots & \cdots & \vdots \\ y_1^{p^n} & \cdots & y_n^{p^n} \end{vmatrix}}{L_n}$$

In the last determinant the p^i -th power is missing. The degrees of the previous elements are p^{i-1}, $\frac{p^n-1}{p-1}$, and $p^n - p^i$ respectively. We note that there are other descriptions for the polynomials above [8].

Theorem 1 (Dickson) $D_n := \mathbb{F}_p[V]^{GL_n} = \mathbb{F}_p[d_{n,0}, \cdots, d_{n,n-1}]$.

Theorem 2 (Mùi) *i)* $H_n := \mathbb{F}_p[V]^{U_n} = \mathbb{F}_p[h_n, \cdots, h_1]$.
ii) $\mathbb{F}_p[V]^{B_n} = \mathbb{F}_p[(h_n)^{p-1}, \cdots, (h_1)^{p-1}]$.

Relations between the generators of rings of invariants are given as follows:

Proposition 3 *[3]* $d_{n,n-i} = \sum_{1 \le j_1 < \cdots < j_i \le n} \prod_{s=1}^{i} \left(h_{j_s}^{p-1}\right)^{p^{n-i+s+j_s}}$.

It is known that any subgroup between B_n and GL_n is conjugate to a parabolic subgroup. Let $N = (n_1, \cdots, n_\ell)$ be a sequence of non-negative integers such that $\sum n_i = n$. Let $P(N,n)$ be the so called parabolic subgroup of GL_n:

$$\begin{pmatrix} GL_{n_1} & * & * \\ 0 & \ddots & * \\ 0 & 0 & GL_{n_\ell} \end{pmatrix}$$

Theorem 4 *(Kuhn and Stong).*

$$\mathbb{F}_p(N,n) := \mathbb{F}_p[V]^{P(N,n)} = \mathbb{F}_p[d_{\nu_i,\nu_i-k_i} \mid 1 \le i \le \ell,\ 1 \le k_i \le n_i,\ \nu_i = \sum_{t=1}^{i} n_t].$$

2 Relations between parabolic and Dickson generators

Since D_n is a subalgebra of $\mathbb{F}_p[V]^{P(N,n)}$, any Dickson generator can be decomposed in terms of generators of the later algebra. We shall describe these relations in this section. For simplicity only the case $N = (n_1, n_2 = n - n_1)$ will be considered. The interested reader can extend those formulas to any number of blocks.

A Dickson generator $d_{n,n-i}$ consists of the sum of all possible combinations of i elements from $\{h_1^{p-1}, \cdots, h_n^{p-1}\}$ in certain p-th exponents. Let $d'_{n,n-i}$ be the polynomial which is given as $d_{n,n-i}$ but elements are from $\{h_{n_1+1}^{p-1}, \cdots, h_n^{p-1}\}$ on the same exponents. Here $n - i \ge n_1$. It is obvious that the new polynomial is a summand of the old one and it will be expressed in terms of old generators.

Example 5 *Let $n = 5$ and $n_1 = 2$.*

i) $d'_{5,4} = h_3^{(p-1)p^2} + h_4^{(p-1)p} + h_5^{(p-1)} = d_{5,4} - d_{2,1}^{p^3}$

ii) $d'_{5,3} = h_5^{(p-1)} h_4^{(p-1)} + h_5^{(p-1)} h_3^{(p-1)p} + h_4^{(p-1)p} h_3^{(p-1)p} = d_{5,3} - d_{2,1}^{p^2} d'_{5,4} - d_{2,0}^{p^3} = d_{5,3} - d_{2,1}^{p^2} d_{5,4} + d_{2,1}^{p^2+p^3} - d_{2,0}^{p^3}$

iii) $d'_{5,2} = h_5^{(p-1)} h_4^{(p-1)} h_3^{(p-1)} = d_{5,2} - d_{2,1}^{p} d'_{5,3} - d_{2,0}^{p^2} d'_{5,4} = d_{5,2} - d_{2,1}^{p} d_{5,3} + d_{2,1}^{p^2+p} d_{5,4} - d_{2,1}^{p+p^2+p^3} + d_{2,0}^{p^3} d_{2,1}^{p} - d_{2,0}^{p^2} d_{5,4} + d_{2,0}^{p^2} d_{2,1}^{p^3}$

Proposition 6 *Let $n = n_1 + n_2$ and $n_2 \geq i \geq 1$. Then*

$$d'_{n,n-i} = d_{n,n-i} + \sum_{t=1}^{i}\sum_{\ell=1}^{t} d_{n,n-i+t} \sum_{\substack{j_1+\ldots+j_l=t\\ 0<j_s\leq n_1}} (-1)^{\ell} d_{n_1,n_1-j_1}^{p^{n_2-i+t}} d_{n_1,n_1-j_2}^{p^{n_2-i+t-j_1}} \ldots d_{n_1,n_1-j_l}^{p^{n_2-i+t-j_1-\ldots-j_{l-1}}}$$

Here $d_{k,k} = 1$.

Proof. First we decompose $d'_{n,n-i}$ in terms of $d'_{n,n-t}$ for $t < i$.

$$d'_{n,n-i} = d_{n,n-i} - \sum_{t=1}^{i} d'_{n,n-i+t} d_{n_1,n_1-t}^{p^{n_2-i+t}}$$

Proof by induction on $1 \leq i \leq n$ and use of Proposition 3.
For the proof of the statement induction is used on $1 \leq i \leq n$. For the general step, each $d'_{n,n-i+t}$ is decomposed (by induction hypothesis) and terms are collected with respect to $d_{n,n-i+c}$'s for $1 \leq c \leq i$.

$$\left[\sum_{t=1}^{c} d_{n_1,n_1-t}^{p^{n_2+t-i}} \sum_{\ell=0}^{c-t}(-1)^{\ell+1} \sum_{j_1+\ldots+j_l=i-t} d_{n_1,n_1-j_1}^{p^{n_2-i+c}} d_{n_1,n_1-j_2}^{p^{n_2-i+c-j_1}} \ldots d_{n_1,n_1-j_l}^{p^{n_2-i+c-j_1-\ldots-j_{l-1}}}\right] d_{n,n-i+c}$$

We have to show that the expression above coincides with the required expression:

$$\sum_{\ell=1}^{c} d_{n,n-i+c} \sum_{j_1+\ldots+j_l=i} (-1)^{\ell} d_{n_1,n_1-j_1}^{p^{n_2-i+t}} d_{n_1,n_1-j_2}^{p^{n_2-i+t-j_1}} \ldots d_{n_1,n_1-j_l}^{p^{n_2-i+t-j_1-\ldots-j_{l-1}}}$$

It is obvious that the last expression contains the previous one. The other direction is shown by considering an individual member of the one above. ∎

Now the next proposition is obvious.

Proposition 7 $\mathbb{F}_p(n_1, n_2) :=\mathbb{F}_p[V]^{P(n_1,n_2)} =\mathbb{F}_p[d_{n_1,j},\, d'_{n,n-i} \,|\, 0 \leq j \leq n_1 - 1$ *and* $1 \leq i \leq n_2]$.

Our last task in this section is to provide relations between parabolic and Dickson invariants.

Example 8 *We continue our last example.* $d_{5,0} = (h_1 \cdots h_5)^{p-1} = d_{2,0}d'_{5,2} = d_{2,0}d_{5,2} - d_{2,0}d_{2,1}^{p}d_{5,3} + d_{2,0}d_{2,1}^{p+p^2}d_{5,4} - d_{2,0}d_{2,1}^{p+p^2+p^3} + d_{2,0}^{1+p^3}d_{2,1}^{p} - d_{2,0}^{1+p^2}d_{5,4} + d_{2,0}^{1+p^2}d_{2,1}^{p^3}$.

$d_{5,1} = (h_2 \cdots h_5)^{p-1} + (h_1 \cdots h_4)^{p(p-1)} + \cdots + (h_1 \cdots h_3)^{p^2(p-1)} h_5^{p-1} = d_{2,1}d'_{5,2} + d_{2,0}^{p}d'_{5,3} = d_{2,1}d_{5,2} - d_{2,1}^{1+p}d_{5,3} + d_{2,1}^{1+p+p^2}d_{5,4} - d_{2,1}^{1+p+p^2+p^3} - d_{2,0}^{p^3}d_{2,1}^{1+p} - d_{2,0}^{p^2}d_{2,1}d_{5,4} + d_{2,0}^{p^2}d_{2,1}^{1+p^3} + d_{2,0}^{p}d_{5,3} - d_{2,0}^{p}d_{2,1}^{p^2}d_{5,4} + d_{2,0}^{p}d_{2,1}^{p^2+p^3} - d_{2,0}^{p+p^3}$.

Proposition 9 *Let* $0 \le i \le n_1 - 1$. *Then*

$$d_{n,i} = \sum_{t \ge \max(0, i-n_2)}^{i} d_{n_1,t}^{p^{i-t}} d'_{n,n_1+i-t}$$

Proof. We use formula 3 and consider each product like being divided in to two parts according to indices of the h_i^{p-1}'s. The first part consists of those h_i^{p-1}'s such that their indices are less or equal than n_1. Then we group monomials according to the second part. ■

3 $\mathbb{F}_p(n_1, n_2)$ as a free module over D_n

3.1 Preliminary

Let p be a prime number and k, s and r_t be natural numbers such that $1 \le t \le s$ and $1 \le r_t < r_{t+1}$.

Definition 10 *Given two sequences consisting of* s *natural numbers* $(k + r_s, ..., k + r_1)$ *and* $(r_s, ..., r_1)$, r_i *as above, the symbol* $\left[{k+r_s,...,k+r_1 \atop r_s,...,r_1}\right]$ *stands for the fraction* $\frac{\prod_{i=1}^{s}(k+r_i)}{\prod_{i=1}^{s} r_i}$. $\left[{k+r_s,...,k+r_1 \atop r_s,...,r_1}\right]$ *is called admissible, if the number of times a particular natural number is a divisor of the* r_t*'s is less or equal than the number of times that particular number is also a divisor of the* $k + r_j$*'s.*

Lemma 11 *If* $\left[{k+r_s,...,k+r_1 \atop r_s,...,r_1}\right]$ *is admissible, then* $\left[{k+r_s,...,k+r_1 \atop r_s,...,r_1}\right]$ *is integral.*

Example 12 *a)* $\begin{bmatrix} 13, & 12, & 11, & 10, & 9, & 8 \\ 6, & 5, & 4, & 3, & 2, & 1 \end{bmatrix}$ *is admissible:* $\{6,3,2\}$, $\{5\}$, $\{4,2\}$, $\{3\}$, $\{2\}$ *and* $\{13\}$, $\{12,6,4,3,2\}$, $\{11\}$, $\{10,5,2\}$, $\{8,4,2\}$, $\{9,3\}$.

$b)$ $\begin{bmatrix} 30, & 28, & 23, & 22, & 20 \\ 12, & 10, & 5, & 4, & 2 \end{bmatrix}$ *is integral but it is not admissible:*
i) $20x22x23x28x30/2x4x5x10x12 = 1771;$
ii) $\{\underline{12}, 6, 4, 3, 2\}$, $\{10, 5, 2\}$, $\{5\}$, $\{2, 4\}$, $\{2\}$ *and* $\{30, 15, 10, 6, 5, 3, 2\}$, $\{28, 14, 7, 4, 2\}$, $\{23\}$, $\{22, 11, 2\}$, $\{20, 10, 5, 4, 2\}$.

Let us reserve the symbol $\left[{k+r \atop r}\right]$ for the sequence $r_t = t$.

Proposition 13 $\left[{k+r \atop r}\right]$ *is admissible for* k *and* $r \geq 1$.

Proof. Using double induction and the well known formula $\binom{k+r}{r} = \binom{k+r-1}{r} + \binom{k+r-1}{r-1}$, the integrality of $\binom{k+r}{r}$ is proved. We assume that $\left[{k+r-1 \atop r-1}\right]$ is admissible. If r is a prime number or divides $(k+r)$, then the statement follows. Otherwise, let $\frac{r}{q}$ be a divisor of r. Then $\frac{r}{q}$ is also a divisor of $m\frac{r}{q}$ for $1 \leq m \leq q$. We need to show that there exist at least q natural numbers $k + r'$ between $k+1$ and $k+r$ which are divisible by $\frac{r}{q}$. Let $k = x\frac{r}{q} + l$ with $0 \leq l \leq \frac{r}{q} - 1$. Then $k + r = (x+q)\frac{r}{q} + l$ and the statement follows. ■

Lemma 14 *If* $\left[{k+r_s,\ldots,k+r_1 \atop r_s,\ldots,r_1}\right]$ *is admissible, then* $\left[{m+k+r_s,\ldots,m+k+r_1 \atop r_s,\ldots,r_1}\right]$ *is also admissible, where* m *is a multiple of* $\mathrm{lcm}(r_s, \ldots, r_1)$.

We shall define a peculiar division in $\left[{k+r \atop r}\right]$ as follows:

Definition 15 *For each element* r_t *of* $(r_s, \ldots, r_1)$ *we let* r_t *divide* $k + r_l$, r_t"\"$k + r_l$, *if* $\frac{k+r_l}{r_t}$ *is integral,* $l \leq t$ *and* l *is maximal with this property.*

Let $I_{exact} = \{r_t \mid r_t | k + r_t\} \subset \{1, \ldots, r\}$. Now we define a partition of $\{1, \ldots, r\}$ according to the given division as follows: For each $r_{t(0)} \in I_{exact}$ let $I_{r_{t(0)}}$ contain all natural numbers l between r and $r_{t(0)}$ such that there is a subsequence $(l = r_{t_l}, r_{t_l-1}, \ldots, r_{t_l-s_l} = r_{t(0)})$ with r_{t_l-s}"\"$k + r_{t_l-s-1}$. Now $\{1, \ldots, r\} = \bigsqcup_{r_{t(0)}} I_{r_{t(0)}}$.

Example 16 *a)* $\left[{23=12+11 \atop 11}\right]$, $I_{exact} = \{6, 4, 3, 2, 1\}$, $I_6 = \{9, 6\}$, $I_4 = \{11, 10, 8, 4\}$, $I_3 = \{5, 3\}$, $I_2 = \{7, 2\}$ *and* $I_1 = \{1\}$. $\begin{bmatrix} 21, & 18 \\ 9, & 6 \end{bmatrix}$, $\begin{bmatrix} 23, & 22, & 20, & 16 \\ 11, & 10, & 8, & 4 \end{bmatrix}$, $\begin{bmatrix} 17, & 15 \\ 5, & 3 \end{bmatrix}$, $\begin{bmatrix} 19, & 14 \\ 7, & 2 \end{bmatrix}$ *are all admissible.*

b) $\left[{29=18+11 \atop 11}\right]$, $I_{exact} = \{9,6,3,2,1\}$, $I_9 = \{9\}$, $I_6 = \{8,6\}$, $I_3 = \{7,3\}$, $I_2 = \{11,4,2\} \cup \{10,2\} \cup \{5,2\}$ *and* $I_1 = \{1\}$. $\begin{bmatrix} 27 \\ 9 \end{bmatrix}$, $\begin{bmatrix} 26, & 24 \\ 8, & 6 \end{bmatrix}$, $\begin{bmatrix} 25, & 21 \\ 7, & 3 \end{bmatrix}$ *are admissible sequences but* $\begin{bmatrix} 29, & 28, & 23, & 22, & 20 \\ 11, & 10, & 5, & 4, & 2 \end{bmatrix}$ *is not even integral.*

Remark 1 *We note that only sequences satisfying certain properties will be considered in this work due to great technicalities.*

Definition 17 *According to division defined above between sequences and for any prime number p, we define* A_{r_t} *to be the natural number*

$$A_{r_t} := \begin{cases} p^{r_t - r_{t-i(t)}} \frac{p^{k+r_{t-i(t)}}-1}{p^{r_t}-1}, r_t \neq r_{t-i(t)} \\ \frac{p^{k+r_t}-p^{r_t}}{p^{r_t}-1}, r_t = r_{t-i(t)} \end{cases}$$

And

$$X\begin{bmatrix} k+r_s, ..., k+r_1 \\ r_s, ..., r_1 \end{bmatrix} := \prod_{t=1}^{s} \frac{p^{k+r_t}-1}{p^{r_t}-1}$$

$$X\begin{bmatrix} k+r \\ r \end{bmatrix} := \prod_{t=1}^{r} \frac{p^{k+t}-1}{p^{t}-1}$$

Remark 2 *Note that* A_i *is the same in* $X\left[{k+r \atop r}\right]$ *and* $X\left[{k+r-1 \atop r-1}\right]$ *for* $i = 1, ..., r-1$.

Proposition 18 $X\left[{k+r_s,...,k+r_1 \atop r_s,...,r_1}\right]$ *is integral if and only if* $\left[{k+r_s,...,k+r_1 \atop r_s,...,r_1}\right]$ *is admissible.*

Proof. It is known that $x^k - 1 = \prod_{d|k} C_d$, where C_d stands for the d-th cyclotomic polynomial. Thus $\prod_{t=1}^{s} \frac{x^{k+r_t}-1}{x^{r_t}-1} = \prod_{t=1}^{s} \frac{\prod_{d|(k+r_t)} C_d}{\prod_{d'|r_t} C_{d'}}$ and the statement follows because of the definition of admissibility. ■

Lemma 19 *Let* $\left[{k+r_s,...,k+r_1 \atop r_s,...,r_1}\right]$ *such that* $r_t | k + r_{t-1}$, $r_1 | k$ *and for each divisor* d *of* $\gcd(r_1, r_2)$
$d | k + r_s$ *or* $\exists t : d \nmid r_t$ *and* $d | k + r_{t-1}$. *Then* $X\left[{k+r_s,...,k+r_1 \atop r_s,...,r_1}\right]$ *is integral.*

Remark 3 *k in the last lemma must satisfy the following conditions: $k = l' \operatorname{lcm}(r_1, ...r_s) + l \operatorname{lcm}(r_1, r_2) - r_1$ such that l' is a non-negative integer and l is a positive integer and $l \operatorname{lcm}(r_1, r_2) - r_1 + r_{i-1} \equiv 0 \mod r_i$. If $r_s = s$, then $k = l \operatorname{lcm}(2, ..., s) + 1$.*

Example 20 *$\left[{19,18,16 \atop 9,8,4}\right]$ is integral although $X\left[{19,18,16 \atop 9,8,4}\right]$ is not integral.*

Proposition 21 *Let $\left[{k+r_s,\ldots,k+r_1 \atop r_s,\ldots,r_1}\right]$ such that the r_t's are as in the last lemma. Then $X\left[{k+r_s,\ldots,k+r_1 \atop r_s,\ldots,r_1}\right] = \sum_{i=0}^{s} \prod_{t=0}^{i} A_{r_t}$. Here $A_{r_i} = p^{r_i - r_{i-1}} \frac{p^{k+r_{i-1}}-1}{p^{r_i}-1}$, $A_{r_i} = p^{r_1} \frac{p^k-1}{p^{r_1}-1}$ and $A_0 = 1$.*

Proof. We use induction on s. $1 + A_{r_1} = \frac{p^{k+r_1}-1}{p^{r_1}-1}$. For the general step: $X\left[{k+r_{s-1},\ldots,k+r_1 \atop r_{s-1},\ldots,r_1}\right] + \prod_{t=0}^{s} A_{r_t} = \prod_{t=1}^{s} \frac{p^{k+r_t}-1}{p^{r_t}-1} \Leftrightarrow \prod_{t=0}^{s} A_{r_t} = X\left[{k+r_{s-1},\ldots,k+r_1 \atop r_{s-1},\ldots,r_1}\right] p^{r_s} \frac{p^k-1}{p^{r_s}-1}$ $\Leftrightarrow p^{r_s} \frac{p^k-1}{p^{r_1}-1} \prod_{t=2}^{s-1} \frac{p^{k+r_{t-1}}-1}{p^{r_t}-1} = \prod_{t=1}^{s} \frac{p^{k+r_t}-1}{p^{r_t}-1}$. ■

The next corollary serves as a collection of relations between generators of D_{n_1}.

Corollary 22 *a) Let $0 \le i \le n_1 - 1$ and $(n_1 - i)$ divides $n - i$. Then $d_{n,i}$ contains $d_{n_1,i}^{A_i+1}$ as a summand. Moreover d'_{n,n_1} contains $d_{n_1,i}^{A_{n_1-i}}$ as a summand.*

b) Let $0 \le i \le n_1 - 1$ and $(n_1 - i) \in I_{r_{t(0)}}$. Then $d_{n,i}$ contains $d_{n_1,i} d_{n_1,n_1-r_{t(0)}}^{A_{r_{t(0)}}}$ as a summand.

c) Let i and v_i such that $0 \le i \le n_1 - 1$, $n - i - v_i = l_i(n_1 - i)$ and $0 \le v_i \le n_1 - i$ where v_i is the smallest with this property. Then $d_{n,i+v_i} d_{n_1,i} - d_{n,i} d_{n_1,i+v_i}$ contains $d_{n_1,i}^{1+A_{n_1-i}}$ as a summand.

d) Let i and t such that $0 \le i < t \le n_1 - 1$, $(n_1 - i)" \nmid "(n - t)$ and $0 \le t' < t$ where $(n_1 - t')"|"(n - t)$ and t' is maximal with this property. Then $d_{n,t} d_{n_1,i} - d_{n,i} d_{n_1,t}$ contains $d_{n_1,i} d_{n_1,t'}^{A_{n_1-t'}}$ as a summand.

Proof. The proof depends on Propositions 9 and 6.
a) Since $(n_1 - i)$ divides $(n - i) = (n - n_1 + n_1 - i)$ and $(n - n_1)$, $(p^{n_1} - p^i)$ divides $(p^n - p^i)$ and this is how A_{n_1-i} has been defined in 17. Moreover, $d_{n,i}$ contains $d_{n_1,i} d'_{n,n_1}$ and d'_{n,n_1} contains $d_{n_1,i}^{A_{n_1-i}}$ (Proposition 6).
b) By definition $r_{t(0)}$ divides $n_2 + r_{t(0)}$, hence $r_{t(0)}$ divides n_2. This implies, as in a), that d'_{n,n_1} contains $d_{n_1,n_1-r_{t(0)}}^{A_{r_{t(0)}}}$.

c) We use last Proposition for $i = i + v_1$ and $t = i$ and Proposition 6 for the decomposition of d'_{n,n_1+v_i}. Then $d'_{n,i+v_i}$ contains $d_{n_1,i}^{\left(\sum_{t=1}^{l_i-1} p^{(v_i+(n_1-i)t)}\right)}$ as a summand. Let us recall definition 17: $A_{n_1-i} = v_i \sum_{t=0}^{l_i-1} p^{(n_1-i)t}$.

d) Let $s = t-t'$ in $d_{n,t}d_{n_1,i}-d_{n,i}d_{n_1,t} = \sum_{s=1}^{t} d'_{n,n_1+s}d_{n_1,t-s}d_{n_1,i}-\sum_{s=1}^{i} d'_{n,n_1+s}d_{n_1,i-s}d_{n_1,t}$, then $d'_{n,n_1+t-t'}$ contains $d_{n_1,t'}^{A_{n_1-t'}-1}$. ■

We shall close this subsection by considering some maps which will let us connect $\mathbb{F}_p(n_1, n_2)$ with $\mathbb{F}_p(n_1 - 1, n_2)$. Those maps have been used by Campbell and Hughes in [1].

Let $\pi_{y_1} : \mathbb{F}_p[y_1, \cdots, y_n] \to \mathbb{F}_p[y_1, \cdots, y_n]$ be the map induced by

$$\pi_{y_1}(y_i) = \begin{cases} 0, & i = 1 \\ y_i, & i > 1 \end{cases}$$

and $sh_{(-1)} : \mathbb{F}_p[y_1, \cdots, y_n] \twoheadrightarrow \mathbb{F}_p[y_1, \cdots, y_{n-1}]$ the one induced by

$$sh_{(-1)}(y_i) = \begin{cases} 0, & i = 1 \\ y_{i-1}, & i > 1 \end{cases}$$

Now let us consider the induced maps on $\mathbb{F}_p(n_1, n_2)$: $sh_{(-1)} \cdot \pi_{y_1}(d_{n_1,i}) = d^p_{n_1-1,i-1}$. It follows that $\mathrm{Im}(sh_{(-1)} \cdot \pi_{y_1}) = \mathbb{F}_p[d^p_{n_1-1,i-1}, d^p_{n-1,j-1} \mid i = 0, ..., n_1-2, j = n_1 - 1, ..., n - 2]$. We need one more map $\varepsilon_{(1)} : \mathrm{Im}(sh_{(-1)} \cdot \pi_{y_1}) \to \mathbb{F}_p(n_1, n_2)$ given by

$$\begin{aligned} \varepsilon_{(1)}(d^p_{n_1-1,i-1}) &= d_{n_1,i} \\ \varepsilon_{(1)}(d^p_{n-1,j-1}) &= d_{n,j} \end{aligned}$$

Lemma 23 *Let $f \in \mathbb{F}_p(n_1, n_2)$, then $\varepsilon_{(1)} \cdot sh_{(-1)} \cdot \pi_{y_1}(f) - f \in (d_{n_1,0})$.*

Here $(d_{n_1,0})$ is the ideal in $\mathbb{F}_p(n_1, n_2)$ generated by the top Dickson generator $d_{n_1,0}$.

Proof. Let $d \in \mathbb{F}_p(n_1, n_2)$ be a non-trivial monomial. We consider two cases. Let $d \notin (d_{n_1,0})$, then d is not divisible by y_1. Otherwise, $d \in (d_{n_1,0})$, since $d \in \mathbb{F}_p(n_1, n_2)$. Thus $d \in (d_{n_1,1}, ..., d_{n,n-1})$ and $d = \prod_{i>0,j\geq n_1} d^{m_i}_{n_1,i}d^{m_j}_{n,j}$. Now, $sh_{(-1)} \cdot \pi_{y_1}(d) = \prod_{i>0,j\geq n_1} d^{pm_i}_{n_1-1,i}d^{pm_j}_{n-1,j}$ and $\varepsilon_{(1)} \cdot sh_{(-1)} \cdot \pi_{y_1}(d) = d$.

Let $d \in (d_{n_1,0}) \Leftrightarrow \pi_{y_1}(d) = 0$. Thus, if $f \in \mathbb{F}_p(n_1, n_2)$ and $g \in \mathrm{Im}(sh_{(-1)} \cdot \pi_{y_1})$ such that $g = sh_{(-1)} \cdot \pi_{y_1}(f)$, then $\varepsilon_{(1)}(g) - f \in (d_{n_1,0})$. ■

3.2 Main Results

Since U_n is a p-Sylow subgroup of GL_n and H_n is a polynomial algebra, $\mathbb{F}_p(n_1, n_2)$ is Cohen-Macaulay [[8], Proposition 8.3.1]. Hence, $\mathbb{F}_p(n_1, n_2)$ is a free module over D_n and we shall provide a free basis for particular choices of n_1 and n_2.

In the sequel $B_A(A')$ stands for a free module basis of the algebra A' over the algebra A and $B'_A(A')$ for a module basis respectively.

Bases are known for H_n and $\mathbb{F}_p[V]^{B_n}$ over D_n.

Theorem 24 *[1], [3]. The set* $B_{D_n}(H_n) = \{h_1^{r_1} \cdots h_n^{r_n} \mid 0 \leq r_i < p^{n-i+1} - 1\}$ *is a free module basis for* H_n *over* D_n.

We extended the last theorem for the parabolic subgroups following the method used in [3].

Theorem 25 *[4]i) The set* $B_{\mathbb{F}_p[V]^{B_n}}(H_n) = \{h_1^{r_1} \cdots h_n^{r_n} \mid 0 \leq r_i < p - 1\}$ *is a free module basis for* H_n *over* $\mathbb{F}_p[V]^{B_n}$.

ii) The set

$$B_{\mathbb{F}_p(N,n)}\left(\mathbb{F}_p[V]^{B_n}\right) = \left\{h_1^{(p-1)r_1} \cdots h_n^{(p-1)r_n} \mid 0 \leq r_i < \tfrac{p^{\nu_s - i+1}-1}{p-1},\ \nu_{s-1} < i \leq \nu_s\right\}$$

is a free module basis for $\mathbb{F}_p[V]^{B_n}$ *over* $\mathbb{F}_p(N, n)$. *Here,* $\nu_i = \sum\limits_{t=1}^{i} n_t$.

We shall make a few remarks concerning the key-points of the proof of theorems above.

i) Let G be one of the groups under consideration. There is a close relation between degrees of generators and the order of the group: $\mathbb{F}_p[V]^G = \mathbb{F}_p[\alpha_1, \cdots, \alpha_n]$, $\prod\limits_1^n |\alpha_i| = 2^n|G|$. The rank of H_n over $\mathbb{F}_p[V]^G$ is $[G : U_n]$.

ii) Let $P(G, t)$ denote the Poincaré series of $\mathbb{F}_p[V]^G$. Note that $|h_i|$ divides $|\alpha_i|$ and hence $P\left(\mathbb{F}_p[V]^G, t\right) / P(H_n, t) = \prod\limits_i \left(1 + t^{|h_i|} + t^{2|h_i|} + \cdots + t^{\left(\frac{|\alpha_i|}{|h_i|}-1\right)|h_i|}\right)$.

The last statement along with proposition 3 provides a strong hint for a set of free basis generators, namely those expressed in the last theorems.

This is not the case for $\mathbb{F}_p(n_1, n_2)$ over D_n as the next example suggests.

Example 26 *Let* $n = 5$ *and* $n_1 = 2$. $|d_{5,0}| = p^5 - 1$, $|d_{2,0}| = p^2 - 1$ *and* $\frac{p^5-1}{p^2-1}$ *is not integral. Hence the method described above does not apply. But* $X\begin{bmatrix}5\\2\end{bmatrix}$

is integral and according to Proposition 21

$$X\begin{bmatrix}5\\2\end{bmatrix} = 1 + A_1 + A_1A_2 = 1 + p^3 + p^2 + p + (p^3 + p^2 + p)(p^3 + p)$$

Proposition 27 *Let* $n_2 = l\,\mathrm{lcm}(2,...,n_1)$. *Then*

$$B_{D_n}(\mathbb{F}_p(n_1,n_2)) = \left\{ \prod_{i=0}^{n_1-1} d_{n_1,i}^{m_i} \,|\, 0 \le m_i \le A_{n_1-i} \right\}$$

is a free module basis for $\mathbb{F}_p(n_1,n_2)$ *over* D_n.

Proof. Let us recall that $A_{n_1-i} + 1 = \sum_{t=0}^{n_2/(n_1-i)} p^{(n_1-i)t}$. The proof depends on the following facts:
i) The hypothesis about n_2 guarantees that $\frac{p^{n-i}-1}{p^{n_1-i}-1}$ is integral for each i.
ii) $d_{n_1,i}^{A_{n_1-i}+1}$ is a summand in the decomposition of $d_{n,i}$ (22-a)).
iii) Mimic the proof of last theorem. ■

We continue our last example.

Example 28 $d_{5,1} = d_{2,1}d_{5,2} - d_{2,1}^{1+p}d_{5,3} + d_{2,1}^{1+p+p^2}d_{5,4} - d_{2,1}^{1+p+p^2+p^3} - d_{2,0}^{p^3}d_{2,1}^{1+p} - d_{2,0}^{p^2}d_{2,1}d_{5,4} + d_{2,0}^{p^2}d_{2,1}^{1+p^3} + d_{2,0}^{p}d_{5,3} - d_{2,0}^{p}d_{2,1}^{p^2}d_{5,4} + d_{2,0}^{p}d_{2,1}^{p^2+p^3} - d_{2,0}^{p+p^3}$. *This equation provides a bound for the top degree of* $d_{2,1}$. *We shall also find bounds for the monomial* $d_{2,0}^{i}d_{2,1}^{j}$.
$d_{5,0} = d_{2,0}d_{5,2} - d_{2,0}d_{2,1}^{p}d_{5,3} - d_{2,0}d_{2,1}^{p+p^2}d_{5,4} - d_{2,0}d_{2,1}^{p+p^2+p^3} + d_{2,0}^{1+p^3}d_{2,1}^{p} - d_{2,0}^{1+p^2}d_{5,4} + d_{2,0}^{1+p^2}d_{2,1}^{p^3}$. *We use the last equation for the monomial* $d_{2,0}d_{2,1}^{p+p^2+p^3}$. $d_{5,1}d_{2,0} - d_{5,0}d_{2,1} = -2\, d_{2,0}^{1+p^3}d_{2,1}^{p+1} + d_{2,0}^{1+p}d_{5,3} - d_{2,0}^{1+p}d_{2,1}^{p^2}d_{5,4} + d_{2,0}^{1+p}d_{2,1}^{p^3+p^2} - d_{2,0}^{1+p+p^3}$. *The following set provides a free basis* $\{ d_{2,0}^{j_0}d_{2,1}^{j_1} \,|\, 0 \le j_0 \le p^3 + p,$ *if* $0 \le j_1 < p^3 + p^2 + p;\ 0 \le j_1 \le p^3 + p^2 + p$ *if* $j_0 = 0\}$. *The cardinality of the last set is* $X\begin{bmatrix}5\\2\end{bmatrix}$.

Proposition 29 *Let* $n_2 = l\,\mathrm{lcm}(2,...,n_1) + 1$. *Then*

$$B_{D_n}(\mathbb{F}_p(n_1,n_2)) = \{1\} \cup \left\{ \prod_{t=i}^{n_1-1} d_{n_1,t}^{m_t} \,|\, 1 \le m_i \le A_{n_1-i} \text{ and } 0 \le m_t < A_{n_1-t}, t > i \right\}_{i=0}^{n_1-1}$$

is a free module basis for $\mathbb{F}_p(n_1,n_2)$ *over* D_n.

Proof. Let us recall that the given hypothesis has been studied in Lemma 21 and the Proposition after as well as the number $\begin{bmatrix} n_2+n_1 \\ n_1 \end{bmatrix}$. Because the cardinality of $B_{D_n}(\mathbb{F}_p(n_1,n_2))$ is the right one, we only have to show that this set is actually a generating set. Let us explain how the claimed set has been deduced. Combining the facts that in this case $t|n_2+t-1$ and Corollary 22 a) and d) we have two families of relations: i) $d_{n_1,i}d_{n_1,n_1-1}^{A_1}$ is a summand in the decomposition of $d_{n,i}$. ii) $d_{n_1,i}d_{n_1,i+t}^{A_{n_1-i-t}}$ is contained in $d_{n,i+t+1}d_{n_1,i}-d_{n,i}d_{n_1,i+t+1}$ for $t\geq 1$. We shall also show that $d_{n,i+t+1}d_{n_1,i}$ and $d_{n,i}d_{n_1,i+t+1}$ have only one monomial in common. So let $d_{n_1,i+t+1}d^m$ be such a monomial in $d_{n,i+t+1}$. Then $k=p^n-p^{n_1}=p^{n_1}(p^{n_2}-1)$. Thus a factor of the degree of possible elements which is not a p-th power must divide $(p^{n_2}-1)$. Because $n_2=l\,\mathrm{lcm}(2,...,n_1)+1$ and that divisor is a product of $(p^{n_1-t}-1)$, we conclude that there is only such element, namely $d_{n_1,n_1-1}^{A_{n_1-1}}$. To prove that the claimed set satisfies the required property we use double induction on, the total degree, $\sum m_i(p^{n_1}-p^i)$, of a typical monomial $\prod_{i=0}^{n_1-1} d_{n_1,i}^{m_i}$ and n_1. For $n_1=1$, this case is an application of the last Proposition. Since

$$d_{n,0}=d_{1,0}d'_{n,1}=d_{1,0}(d_{n,1}+\sum_{t=1}^{n-1}(-1)^t d_{n,1+t}d_{1,0}^{p+...+p^t})$$

$d_{1,0}^{1+p+...+p^t}$ decomposes with respect to the given basis. Let $f=\prod_{i=0}^{n_1-1} d_{n_1,i}^{m_i}$. Then $\varepsilon_{(1)}\cdot sh_{(-1)}\cdot\pi_{y_1}(f)-f\in(d_{n_1,0})$ (please see lemma 23). Let $g^p=sh_{(-1)}\cdot\pi_{y_1}(f)$ and we decompose g in $\mathbb{F}_p(n_1-1,n_2)$ by induction and the remark 2. Let us apply $\varepsilon_{(1)}$ on the p-th power of the last decomposition and call that element $g(n)$. Then $g(n)$ fulfils the requirements of our basis. Thus $f-g(n)=d_{n_1,0}h$ and by induction h can be decomposed as h'. Finally, $d_{n_1,0}h'$ decomposes according to our relations regarding $d_{n_1,0}^{A_{n_1}+1}$ and $d_{n_1,0}d_{n_1,i}^{A_{n_1-i}}$. ■

We combine the last two Propositions by applying our method in the case $n=11$ and $n_1=5$.

Example 30 *Let $n=11$ and $n_1=5$. We compute A_i for $i=5,..,1$. $I_1=\{1\}$, $I_2=\{5,4,2\}$ and $I_3=\{3\}$. $A_5=p(p^5+1)$, $A_4=p^2(p^4+1)$, $A_3=p^6+p^3$, $A_2=p^6+p^4+p^2$, and $A_1=p^6+...+p$. Now a free basis $B_{D_{11}}(\mathbb{F}_p(5,6))$ follows:*
$\{d_{5,2}^{m_2}d_{5,4}^{m_4}|0\leq m_2\leq A_3, 0\leq m_4\leq A_1\}\cup\{d_{5,2}^{m_2}d_{5,3}^{m_3}d_{5,4}^{m_4}|0\leq m_2\leq A_3, 0\leq m_3\leq A_2, 0\leq m_4\leq A_1\}\cup\{d_{5,1}^{m_1}d_{5,2}^{m_2}d_{5,3}^{m_3}d_{5,4}^{m_4}|1\leq m_1\leq A_4, 0\leq m_2\leq A_3, 0\leq m_3\leq$

$A_2, 0 \leq m_4 \leq A_1\} \cup \{d_{5,0}^{m_0} d_{5,1}^{m_1} d_{5,2}^{m_2} d_{5,3}^{m_3} d_{5,4}^{m_4} | 1 \leq m_0 \leq A_5, 0 \leq m_1 \leq A_4, 0 \leq m_2 \leq A_3, 0 \leq m_3 \leq A_2, 0 \leq m_4 \leq A_1\}$.

Theorem 31 *The set* $B' = \left\{ \prod_{i=0}^{n_1-1} d_{n_1,i}^{m_i} \,|\, 0 \leq m_i \leq A_i \right\}$ *is a module basis for* $\mathbb{F}_p(n_1, n_2)$ *over* D_n.

Here A_i is as in definition 17.

Proof. Firstly, we recall that $d_{n_1,i}^{A_{n_1-i+1}}$ decomposes with respect to the claimed basis because of Corollary 22. Let $f = \prod_{i=0}^{n_1-1} d_{n_1,i}^{m_i}$ be a typical element. We use double induction on the total degree of f and n_1. We follow the proof of Proposition 29.

Let $m_0 = 0$, then $sh_{(-1)} \cdot \pi_{y_1}(f) = \prod_{i=0}^{n_1-2} d_{n_1-1,i-1}^{pm_i} = \left(\prod_{i=0}^{n_1-2} d_{n_1-1,i-1}^{m_i} \right)^p = \left(\sum_I f_I \right)^p$ with respect to $B'_{D_{n-1}}(\mathbb{F}_p(n_1 - 1, n_2))$. Now, $sh_{(-1)} \cdot \pi_{y_1}(f) = \sum_I f'_I$ with respect to B'. Thus $f - \sum_I f'_I = d_{n_1,0}h$ and induction hypothesis can be applied.

If $m_0 > 0$, two cases should be considered: i) $A_{n_1} + 1 \geq m_0$ and ii) $A_{n_1} \leq m_0$. For i) we use induction on $\prod_{i=1}^{n_1-1} d_{n_1,i}^{m_i}$ and for ii) the relation $d_{n_1,0}^{A_{n_1}+1} = d_{n_1,0} d_{n,i(0)} - d_{n_1,i(0)} d_{n,0} -$ "*others*". ■

4 The transfer between parabolic subgroups

The main results in this section are Theorems 41 and 44. Let us recall part of material which has been appeared in [4].

Let H be a subgroup of a finite group G, then the inclusion $i : H \hookrightarrow G$ induces the transfer map $tr^* : H^*(H,\mathbb{F}_p) \longrightarrow H^*(G,\mathbb{F}_p)$ given by $tr^*(u) = |G : H|^{-1} \sum_{g \in G/H} gu$. If $W_G(H)$ is the Weyl subgroup, then the inclusion above induces $i^* : H^*(G,\mathbb{F}_p) \longrightarrow H^*(H,\mathbb{F}_p)^{W_G(H)}$.

$\sum_{p^n}$ acts on V and if we regard $(\mathbb{F}_p)_i$ as the subgroup of translations in the i-th component, then $\sum_{p^n,p} := (\mathbb{F}_p)_1 \int \cdots \int (\mathbb{F}_p)_n$ is a p-Sylow subgroup of $\sum_{p^n}$. We can regard $\mathbb{F}_p{}^n$ as the subgroup of all translations of V in $\sum_{p^n}$. The Weyl subgroups of $\mathbb{F}_p{}^n$ in $\sum_{p^n;p}$, $\sum_{p^{n_1}} \int \sum_{p^{n_2}}$ and $\sum_{p^n}$ respectively are the upper triangular group U_n, $P(n_1, n_2)$ and the general linear group $GL(n,p)$. The induced inclusion $W_{\sum_{p^n,2p}}(\mathbb{F}_p^n) \to W_{\sum_{p^{n_1}} \int \sum_{p^{n_2}}}(\mathbb{F}_p{}^n) \to W_{\sum_{p^n}}(\mathbb{F}_p^n)$ induces $H^*(\mathbb{F}_p^n, \mathbb{F}_p)^{U_n} \xrightarrow{\tau^*} H^*(\mathbb{F}_p^n, \mathbb{F}_p)^{P(n_1,n_2)} \xrightarrow{\tau^*} H^*(\mathbb{F}_p^n, \mathbb{F}_p)^{GL(n,p)}$ given by $\tau^*(f) = \sum_{g \in G/H} gf$. Here f is a U_n or $P(n_1, n_2)$-invariant polynomial in $\mathbb{F}_p[V]$.

Here V is a $\mathbb{F}_pG$-module. In our case the transfer is surjective and $\mathbb{F}_p[V]^{GL(n,p)}$ is a direct summand.

The following diagram is commutative [5].

$$\begin{array}{ccccc} H^*\left(\sum_{p^n,p}\right) & \xrightarrow{\tau^*} & H^*\left(\sum_{p^{n_1}} \int \sum_{p^{n_2}}\right) & \xrightarrow{\tau^*} & H^*\left(\sum_{p^n}\right) \\ \uparrow & & \uparrow & & \uparrow \\ H^*(\mathbb{F}_p^n, \mathbb{F}_p)^{U_n} & \xrightarrow{\tau^*} & H^*(\mathbb{F}_p^n, \mathbb{F}_p)^{P(n_1,n_2)} & \xrightarrow{\tau^*} & H^*(\mathbb{F}_p^n, \mathbb{F}_p)^{GL(n,p)} \end{array} \quad (1)$$

Campbell and Hughes ([1]) have studied the transfer for the case:

$$\tau^* : H_n \to D_n$$

We recall from [1] a set of coset representatives for GL_n over U_n.

$\mathbb{F}_p^n$ is the $(p^n - 1)$-st cyclotomic field over $\mathbb{F}_p$ and let σ_n be a primitive $(p^n - 1)$-st root of unity over $\mathbb{F}_p$ of an irreducible factor of the $(p^n - 1)$-st cyclotomic polynomial $x^{p^n-1} - 1$. Then $\mathbb{F}_p^{n*} =< \sigma_n >$. Let $\Phi : \mathbb{F}_p^{n*} \cong V - \{\bar{0}\} \to < \sigma_n >$ be the natural correspondence. σ_n acts linearly on V as follows: $\sigma_n \bar{0} = \bar{0}$ and $\sigma_n u = \sigma_n^{k+1}$ where $\Phi(u) = \sigma_n^k$. Note that the fixed point set of σ_n^m is the zero vector set.

Let $W^{n-m+1} = \langle y_{n-m+1}, \cdots, y_n \rangle$, then $(W^{n-m+1})^* = \langle \sigma_m \rangle$ as a multiplicative group and we denote the corresponding linear transformation again by the same symbol as above.

Theorem 32 *[1]The set* $\{\sigma_n^{i_n} \cdots \sigma_1^{i_1} \mid 0 \le i_m < p^m - 1\}$ *is a set of left coset representatives for* GL_n *over* U_n.

We generalize the previous theorem for parabolic subgroups. Let $\Im_n(p-1)$ be the set of coset representatives of GL_n over B_n:

$$\{< \sigma_n > / < \sigma_n^{l_{p-1}} > \text{ such that } | < \sigma_n^{l_{p-1}} > | = p - 1\}$$

Note that this set is induced by $\mathbb{F}_p^{n*} \cong < y_1, ..., y_n > -\{\bar{0}\}$. $|\Im_n(p-1)| = \frac{p^n-1}{p-1}$.

Theorem 33 *i) The set* $\{\sigma_n^{i_n}\cdots\sigma_1^{i_1} \mid 0 \le i_m < p-1\}$ *is a set of left coset representatives for* B_n *over* U_n*. Here* $<\sigma_i>\cong<y_i>-\{\bar{0}\}$.

ii) The set $\{\sigma_n^{i_n}\cdots\sigma_1^{i_1} \mid 0 \le i_m < \frac{p^m-1}{p-1}\}$ *is a set of left coset representatives for* GL_n *over* B_n*. Here* $\sigma_m^{i_m} \in \Im_m(p-1)$ *induced by* $< y_{n-m+1}, \cdots, y_n > -\{\bar{0}\}$.

iii) Let $N = \{n_1, \cdots, n_\ell\}$ *and* $\nu_t = n_1 + \cdots + n_t$*. The set* $\{\prod_{t=1}^{\ell} \sigma_{1,t}^{i_{1,t}} \cdots \sigma_{n_t,t}^{i_{n_t,t}} \mid 1 \le s \le n_t,\ 0 \le i_{s,t} < \frac{p^s-1}{p-1}\}$ *is a set of left coset representatives for* $P(N,n)$ *over* B_n*. Here* $\sigma_{k_t,t}^{i_{k_t,t}} \in \Im_{k_t,t}(p-1)$ *induced by* $< y_{\nu_t-k_t+1}, \cdots, y_{\nu_t} > -\{\bar{0}\}$.

Proof. We use the following embedding:

$$\begin{pmatrix} 1 & 0 \\ 0 & GL_k \end{pmatrix} \hookrightarrow \begin{pmatrix} GL_{k+1} \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & 1 & 0 \\ 0 & \cdots & 0 & GL_{n_\ell} \end{pmatrix} \hookrightarrow \begin{pmatrix} GL_{n_{\ell-1}} & 0 \\ 0 & GL_{n_\ell} \end{pmatrix}$$

We recall that $|GL_n| = \prod_{i=1}^{n} p^n - p^{i-1} = (p^n-1)p^{n-1}|GL_{n-1}|$. The idea is to construct a set of representatives for the last block and to extend it step by step until we cover the whole group.

i) We use induction. Let $C(B_{n-1}/U_{n-1})$ be a set of representatives for B_{n-1} over U_{n-1}. Let σ_n be a primitive element in $\langle y_1\rangle^*$. This element has order $p-1$, acts linearly and moves every element of $\langle y_1\rangle$. Since U_n fixes y_1, $\{\sigma_n^{i_n} \mid 0 \le i_n < p-1\}C(B_{n-1}/U_{n-1}) = C(B_n/U_n)$.

ii) Let $C(GL_{n-1}/B_{n-1})$ be a set of representatives for GL_{n-1} over B_{n-1}. Let σ_n^i be an element in $\Im_n(p-1)$. This element has order p^n-1, acts linearly and moves every element of $\langle y_1, \cdots, y_n\rangle$. Moreover, $\sigma_n^i y_1^{p-1} \ne y_1^{p-1}$. Since B_n fixes (y_1^{p-1}), $\Im_n(p-1)C(GL_{n-1}/B_{n-1}) = C(GL_n/B_n)$.

iii) We only have to show that

$$C(\mathbb{F}_p(n_1,n_2)/B_{n_1+n_2}) = C(GL_{n_1}/B_{n_1})C(GL_{n_2}/B_{n_2})$$

and then the argument follows by induction on the number of blocks. To prove the claim we use induction on n_1. For $n_1 = 2$, $\Im_{1,1}(p-1) = \{e\}$ and $\Im_{2,1}(p-1) = \{\sigma_{2,1}^i \mid 0 \le i < p+1\}$. Now the claim follows. ∎

The method used above to determine coset representatives for $\mathbb{F}_p(n_1, n_2)$ over B_n can not be applied for GL_n over $\mathbb{F}_p(n_1, n_2)$ for the same reasons as in a free module basis construction of the appropriate rings of invariants.

Proposition 34 *Let* $GL_n = \coprod_{i=1}^{|GL_n:B_n|} \sigma_i B_n$ *and* $\mathbb{F}_p(n_1, n_2) = \coprod_{j=1}^{|\mathbb{F}_p(n_1,n_2):B_n|} \tau_j B_n$ *as constructed before. Then* $GL_n = \coprod_{k=1}^{|GL_n:\mathbb{F}_p(n_1,n_2)|} u_k B_n$ *where* $u_k = \sigma_{i(k)}$ *such that* $\sigma_{i(k)} \notin \mathbb{F}_p(n_1, n_2)$ *and* $\sigma_{i(k)} \notin \sigma_{i(j)}\mathbb{F}_p(n_1, n_2)$ *for* $k > j$.

Definition 35 *Let the trace over the matrix* σ_m *be defined by* $Tr_m(-) = (p^m - 1)^{-1} \sum_{i=1}^{p^m-1} \sigma_m^i(-)$ *and* $Tr_{m,t}^{(p-1)}(-) = (\frac{p^m-1}{p-1})^{-1} \sum_{i=0}^{\frac{p^m-1}{p-1}} \sigma_{m,t}^{(p-1)i}(-)$.

Then $\tau^*(f) = \prod_m Tr_m(f)$ for H_n over G and $\tau^*(f) = \prod_t \prod_m Tr_{m,t}^{(p-1)}(f)$ for B_n over G. Here G is $\mathbb{F}_p(N, n)$ or GL_n.

Because of the properties of the matrices σ_m and the generators for D_n, $\mathbb{F}_p(N, n)$, B_n, and H_n we conclude:

$$\tau^*(h_1^{i_1} \cdots h_m^{i_m} \cdots h_n^{i_n}) = Tr_n(h_1^{i_1} \cdots Tr_{n-m+1}(h_m^{i_m} \cdots Tr_1(h_n^{i_n}) \cdots) \cdots)$$

Definition 36 *Let* $b_i = h_i^{p-1}$ *and* $TR_{i,t} = Tr_{i,t}^{(p-1)}$.

$$\tau^*(b_1^{i_1} \cdots b_{n_1}^{i_{n_1}} b_{n_1+1}^{i_{n_1+1}} \cdots b_{\nu_{\ell-1}+1}^{i_{\nu_{\ell-1}+1}} \cdots b_{\nu_\ell}^{i_{\nu_\ell}}) = \prod_t TR_{n_t,t}(b_{\nu_{t-1}+1}^{i_{\nu_{t-1}+1}} \cdots TR_{1,t}\left(b_{\nu_t}^{i_{\nu_t}}\right) \cdots).$$

Let $\xi : H_n \longrightarrow D_n$ be the natural $\mathbb{F}_p$-epimorphism. Campbell and Hughes have shown that the map ξ is actually the induced transfer map between U_n and GL_n-invariants. We showed that the same is true for $\xi : H_n \longrightarrow \mathbb{F}_p(n_1, n_2)$ in [3].

Next we consider the map above for parabolic subgroups: $\xi : \mathbb{F}_p(n_1, n_2) \longrightarrow D_n$.

The next lemma plays a key role in the proof of the main theorem of this section.

Lemma 37 $\sum_{u \in V} u^r \begin{cases} = 0 & if\ r \neq 0 \mod(p^n - 1) \\ \neq 0 & if\ r = 0 \mod(p^n - 1) \end{cases}$ *in* $\mathbb{F}_p[V]$.

Proof. Let $u \in V^*$. Since $|V^*| = (p-1)(p^{n-1}+\cdots+1)$, we write its elements as follows:

$\{a_i(y_i+v_i) \mid a_i \in \mathbb{F}_p{}^*,\ v_i \in < y_{i-1}, \cdots, y_1 >\}$. We use induction on n. For $n=1$, $\sum a^{(p-1)k}y_1^{(p-1)k} = (p-1)y_1^{(p-1)k}$. If $r \neq (p-1)k$, then $\sum a^r = 0$. Let $r = 0 \mod(p^n-1)$, then $a_i^r = 1$. Hence, $\sum_{a_n} a_n^r(y_n+v_n)^r = (p-1)(y_n+v_n)^r$. $\sum_{v_n}(y_n+v_n)^{(p^n-1)} = \sum_{v_n}\sum y_n^{p^n-1-t}v_n^t$. The last sum contains the term $\sum_{v_n} y_n^{p^n-p^{n-1}}v_n^{p^{n-1}-1}$ which is non-zero by induction. The second claim follows. For the first claim, if $r \neq 0 \mod(p-1)$, then $\sum_{a_i} a_i^r(y_i+v_i)^r = 0$. Otherwise, $\sum_{a_i} a_i^r(y_i+v_i)^r = (p-1)(y_i+v_i)^r$. Now, $\sum_{u\in V} u^r = (p-1)\sum_{i,v_i}(y_i+v_i)^r = \sum_{v_n}(p-1)(y_n+v_n)^r + \nu_n^r$. Let $y_n = 0$, then our sum is zero and hence divisible by y_n. On the other hand, this sum is a GL_n-invariant. Thus, this sum is divisible by $d_{n,0}$. ■

Remark 4 *Let $r = (p-1)k \neq 0 \mod(p^n-1)$. By lemma above, $\sum_{u\in V} u^r = (p-1)\sum_{i,v_i}(y_i+v_i)^r = 0$. Hence $\sum_{i,v_i}(y_i+v_i)^{(p-1)k} = 0$.*

Lemma 38 *$TR_{n_t-m+1,t}(b^{i_m}_{\nu_{t-1}+m}) = 0$ for $b^{i_m}_{\nu_{t-1}+m} = h^{(p-1)i_m}_{\nu_{t-1}+m}$ in $B_{\mathbb{F}_p(N,n)}\left(\mathbb{F}_p[V]^{B_n}\right)$.*

Proof. By definition, $TR_{n_t-m+1,t}(b^{i_m}_{\nu_{t-1}+m}) = \sum_{s=0} \sigma^s_{n_t-m+1,t}(b^{i_m}_{\nu_{t-1}+m})$. Here $0 \leq i_m < \frac{p^m-1}{p-1}$, $1 < m \leq n_t$. For the rest of the proof, let us simplify the coefficients $\sigma^s_{n_t-m+1} \rightarrow \sigma^s_m$ and $b^{i_m}_{\nu_{t-1}+m} \rightarrow b^{i_m}_m$. We recall that σ^s_m is a primitive element in $< y_{\nu_{t-1}+m}, \cdots, y_{\nu_t} >^*$ and fixes y_t for $t < \nu_{t-1}+m$. Recall that $h_m(y_m) = y_m^{p^{m-1}} + \sum(-1)^i d_{m-1,i}y_m^{p^i}$, h_m is linear in y_m. Now, we evaluate the sum in the argument. $\sum_{s=0} \sigma^s_m(h_m^{(p-1)i_m}) = \sum_{s=0}\left(h_m^{(p-1)i_m}(\sigma^s_m y_m)\right) = \sum_{i,u_i}\left(h_m^{(p-1)i_m}(y_i+u_i)\right)$. Here $i = \nu_{t-1}+m, ..., \nu_t$ and $u_i \in < y_{\nu_{t-1}+m}, \cdots, y_i >$. The last sum is zero, because of remark 4. ■

A direct application of the last lemma and remark 35 is the next theorem.

Theorem 39 *i) Let $h \neq 1$ be an element of the set $B_{\mathbb{F}_p[V]^{B_n}}(H_n)$. Then $\tau^*(h) = 0$.*

ii) Let $h(p-1) \neq 1$ be an element of the set $B_{\mathbb{F}_p(N,n)}\left(\mathbb{F}_p[V]^{B_n}\right)$. Then $\tau^*(h(p-1)) = 0$.

The next theorem follows from last.

Theorem 40 *i) Let* $\xi : H_n \longrightarrow \mathbb{F}_p[V]^{B_n}$ *be the natural* $\mathbb{F}_p$*-epimorphism. Then* $\xi = \tau^*$.

ii) Let $\xi : \mathbb{F}_p[V]^{B_n} \longrightarrow \mathbb{F}_p(N, n)$ *be the natural* $\mathbb{F}_p$*-epimorphism. Then* $\xi = \tau^*$.

The obvious step at this point is to extend the statements above to $\tau^* : \mathbb{F}_p(n_1, n_2) \rightarrow D_n$. Let B denote the set $B_{D_n}(\mathbb{F}_p(n_1, n_2))$.

Theorem 41 *Let* $\xi : \mathbb{F}_p(n_1, n_2) \longrightarrow D_n$ *be the natural* $\mathbb{F}_p$*-epimorphism with respect to basis* B*. Then* $\xi = \tau^*$.

Proof. The statement follows from the following commutative diagram

$$\begin{array}{ccc} \mathbb{F}_p[V]^{B_n} & \overset{\tau=\xi}{\rightarrow} & \mathbb{F}_p(n_1, n_2) \\ & \searrow^{\tau=\xi} & \downarrow \\ & & D_n \end{array}$$

Here τ and ξ refer to the right map. ■

Kuhn and Priddy have examined a multiplicative property of the transfer between certain subgroups of the symmetric group in [5]. We examine an analogy between certain rings of invariants.

Lemma 42 *Let* h^I *be an element of the set* $B_{D_n}(H_n)$*. Then* $\exists h^J$*, an other basis element, such that* $\tau^*(h^{I+J}) \neq 0$.

Proof. We provide two different types of elements.
i) Let $I = (i_1, ..., i_n)$, we define $J = (j_1, ..., j_n)$ such that

$$j_t = \begin{cases} p^{n-t+1} - 1 - i_t, i_t \neq 0 \\ 0, i_t = 0 \end{cases}$$

Claim: $\tau^*(h^{I+J}) = \prod_{t=0}^{n-1} d_{n,t}^{\varepsilon_t}$ where $\varepsilon_t = 1$ if $i_t \neq 0$ and 0 otherwise. We proceed by induction on the non-zero indexes of I. It is known that $h_t^{p^{n-t+1}-1} = d_{n,t-1}+$"*others*" ([3]). Let us recall that this is the way the basis $B_{D_n}(\mathbb{F}_p[V]^{B_n})$ was constructed. Moreover, $\tau^*(f) = 0$ for all monomials f involved in

"*others*". Thus $h^{I+J} = \prod_{t=0}^{n-1} d_{n,t}^{\varepsilon_t} +$ "*others*" and $\tau^*(f) = 0$ for all monomials f involved in "*others*".
ii) There also exists an other element $h^{J'}$ of degree less or equal than the degree of h^J defined above. Let $l = \max\{t | i_t \neq 0\}$ and $\{i_{s_1}, ..., i_{s_r} = l\}$ be the non-zero elements of I. Define

$$j'_t = \begin{cases} p^{n-l+1} - 1 - i_l \\ p^{n-i_{s_t}+1} - 1 - i_{s_t} \text{ for } t < r, i_{s_t} > p^{i_{s_{t+1}} - i_{s_t}}(p^{n-i_{s_{t+1}}+1} - 1) \\ p^{i_{s_{t+1}} - i_{s_t}}(p^{n-i_{s_{t+1}}+1} - 1) - i_{s_t} \text{ for } t < r, 0 < i_{s_t} \leq p^{i_{s_{t+1}} - i_{s_t}}(p^{n-i_{s_{t+1}}+1} - 1) \\ 0 \text{ for } i_{s_t} = 0 \text{ or } t > l \end{cases}$$

Then $\tau^*(h^{I+J}) \neq 0$. This is because
$h_{i_{s_{t+1}}}^{p^{n-i_{s_{t+1}}+1}-1} = d_{n, i_{s_{t+1}}-1} - h_{i_{s_t}}^{p^{i_{s_{t+1}}-i_{s_t}}(p^{n-i_{s_{t+1}}+1}-1)} +$ "*others*". ■

Proposition 43 *Let* $h = \sum_{h^I \in B_{D_n}(H_n)} h^I$, *then* $\tau^*(hf) \neq 0$ *for any non-zero element of* H_n.

Proof. We apply last lemma for f a monomial of the form $h'd$ where h' is a basis element and $d \in D_n$. Then we extend linearly for any element of H_n. ■

Remark 5 *Lemma and Proposition above extend easily to the case* $B_{D_n}\left(\mathbb{F}_p[V]^{B_n}\right)$.

Theorem 44 *Let* $\tau^* : \mathbb{F}_p[V]^G \to D_n$ *be the transfer map, where* $G = U_n$ *or* B_n. *Let* $x, y \in H_n$ *or* $\mathbb{F}_p[V]^{B_n}$. *Then* $\tau^*(xy) = \tau^*(x)\tau^*(y)$ *for all* y *iff* $x \in D_n$.

Proof. Let $\tau^*(x) = z$ and $\tau^*(y) = w$. Now decompose x and y with respect to the given basis, then $x = z + h_x$ and $y = w + h_y$. Hence $\tau^*(xy) = zw + \tau^*(h_x h_y)$. The last implies that $\tau^*(h_x h_y) = 0$ and $\tau^*(h_x) = 0$. We shall show that $h_x = 0$. Assume that $h_x \neq 0$. Choose $h_y = \sum_{h^I \in B_{D_n}(H_n)} h^I$ then $h_x h_y = d_{xy} + h_{xy}$ and $\tau^*(h_{xy}) = 0$. Thus $\tau^*(h_x h_y) = d_{xy}$. ■

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